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Landau analog levels for dipoles in non-commutative space and phase space

Landau analog levels for dipoles

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Abstract We investigate the analog of Landau quantization, for a neutral polarized particle in the presence of homogeneous electric and magnetic external fields, in the context of non-commutative quantum mechanics. This particle, possessing electric and magnetic dipole moments, interacts with the fields via the Aharonov–Casher and He–McKellar–Wilkens effects. For this model we obtain the Landau energy spectrum and the radial eigenfunctions of the non-commutative space coordinates and non-commutative phase space coordinates. Also we show that the case of non-commutative phase space can be treated as a special case of the usual non-commutative space coordinates.

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1 Introduction

Several topological and geometrical effects can be realized within the framework of the quantum dynamics of charged and neutral particles in the presence of electric and magnetic fields. In 1959, Aharonov and Bohm demonstrated that electromagnetic fields affect the state of matter even in spatial regions where the field is equal to zero [1]. This effect occurs due to the presence of the vector potential in the volume where the particle moves. In other words, in quantum mechanics, in a multiply-connected part of space where there are no fields, the physical properties of the system still depend on the potential, in contrast with classical physics. Within the Aharonov–Bohm effect, a quantum charged particle circulating around a magnetic flux line acquires a quantum topological non-dispersive phase in its wave function. This effect was observed experimentally by Chambers [2, 3].

Years later, in 1984, Aharonov and Casher demonstrated that the wave function of a neutral particle that possesses a non-zero magnetic dipole moment, moving through a non-simply-connected force-free region, is affected by an electric field in a way similar to the Aharonov–Bohm effect [4]. In the Aharonov–Casher effect, a quantum neutral particle with non-zero magnetic dipole moment, moving around and parallel to a charged wire, receives a quantum geometrical non-dispersive phase for the wave function. This effect was observed in a neutron interferometer [5] and in a neutral atomic Ramsey interferometer [6] experiment. Further, He and McKellar in 1993 [7] and Wilkens [8] independently in 1994 predicted the existence of a quantum phase acquired by the wave function of a neutral particle which possesses a non-zero electric dipole moment, while it is circulating around and parallel to a line of magnetic monopoles. A simple practical experimental setup to test the He–McKellar–Wilkens effect, without the inconvenience of magnetic monopoles, was proposed by Wei et al. [9]. In this setup, an electric dipole moment is induced for the neutral particle by the electric field of a charged wire and a uniform magnetic field applied with a non-trivial angle to the electric field. Two other experimental schemes for the He–McKellar–Wilkens phase are proposed by Dowling et al. [10], as well as an unified description of all three phenomena, and they have studied a new effect, that is, the dual Aharonov–Bohm effect. The dual Aharonov–Bohm phase can be calculated in the quantum dynamics of a magnetic monopole in the presence of an electric solenoid [11].

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In 1930, Landau showed that a charged particle moving in an homogeneous magnetic field displays quantized energy levels [12]. The Landau levels have a remarkable role in the study of several problems in physics, e.g. the quantum Hall effect [13], different two-dimensional surfaces [14–16], anyons' excitations in a rotating Bose–Einstein condensate [17, 18], and others like analog levels for dipoles. Ericsson and Sjöqvist developed an analog of Landau quantization for neutral particles in the presence of an external electric field [19]. The idea is based on the Aharonov–Casher effect in which neutral particles may interact with an electric field via a non-zero magnetic dipole moment. In the same way, we have developed an analog of Landau quantization (to which we will further refer as Landau-like quantization) for neutral particles that possess a non-zero electric dipole moment, making use of the He–McKellar–Wilkens effect [20]. To solve the problem of magnetic monopoles, we have proposed the study of an analog of Landau quantization in quantum dynamics of an induced electric dipole in the presence of crossed electric and magnetic fields [21].

Recently, non-commutative space theories, motivated by string theory [22, 23], have attracted interest in several areas of physics [24], such as quantum gravity [25], M-theory [26], and studies of the quantum Hall effect [27–29]. In quantum mechanics, a large number of problems has been investigated in the case of non-commutative space [30] and non-commutative phase space. Some important results are related to geometric phases, such as the Aharonov–Bohm effect [31–35], the Aharonov–Casher effect [36, 37], the Berry quantum phase [38, 39], Landau levels [40–42] and other ones involving the dynamics of dipoles [43]. In a recent paper, we have analyzed the quantum geometrical phase effect for a quantum neutral particle with permanent magnetic and electric dipole moments in the presence of external magnetic and electric fields, proposed by Anandan [44], within the context of non-commutative space and phase space quantum mechanics [45].

The aim of this work is the study of Landau-like energy levels for neutral particles possessing non-zero magnetic and electric dipole moments, in the presence of homogeneous electric and magnetic external fields in the context of non-commutative quantum mechanics. We calculate the corrections to Landau-like energy levels generated due to the non-commutativity of the space and phase space coordinates.

The paper is organized as follows. In Sect. 2 we present a review of the standard Landau quantization for a charged particle moving in a homogeneous external magnetic field [12]. In Sect. 3 we give a brief review of the Landau-like quantization for magnetic and electric dipoles in the presence of external magnetic and electric fields [19, 20]. In Sect. 4 a general overview of non-commutative quantum mechanics is presented [48]. In Sects. 5, 6, 7 and 8 we investigate the Landau-like effects in non-commutative space and phase space. Finally, in Sect. 9 we present our conclusions.

2 Charged particle in a homogeneous magnetic field

It is well known that one can easily obtain quantized energy levels for a charged particle, with charge $-e$, moving in the x–y plane and exposed to an external homogeneous magnetic field directed along the z-axis, $\boldsymbol{B} = B_0\hat{e}_z$. These energy levels are called Landau levels [12]. The Hamiltonian of the system is given by

$$H = \frac{1}{2m}(\hat{p} - e\boldsymbol{A})^2, \tag{1}$$

where $\boldsymbol{A}$ is the vector potential, $\boldsymbol{B} = \nabla \times \boldsymbol{A}$, and $\hat{p}$ is the linear momentum, $\hat{p} = -\mathrm{i}\nabla$ (we use the natural unit system $\hbar = c = 1$). We choose the symmetric gauge

$$\boldsymbol{A} = \frac{B_0}{2}(-y, x, 0) = \frac{B_0}{2} r\hat{e}_\phi, \tag{2}$$

where, in cylindrical coordinates, $r^2 = x^2 + y^2$ and $\hat{e}_\phi$ is the unitary vector oriented in ϕ-direction. Hence, from (1), we can write the Schrödinger equation, making use of the cylindrical symmetry of the system, in the form

$$-\frac{1}{2m}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\phi^2}\right] - \frac{\mathrm{i}\omega}{2}\frac{\partial\psi}{\partial\phi} + \frac{m\omega^2}{8}r^2\psi = \mathcal{E}\psi, \tag{3}$$

where

$$\omega = \frac{eB}{m} \tag{4}$$

is the cyclotron frequency. Solving (3), we find the Landau levels, given by

$$\mathcal{E} = \left(n + \frac{|\ell| + \ell + 1}{2}\right)\omega, \tag{5}$$

where ℓ is an integer number related to the wave function periodicity, in the form

$$\psi = \mathrm{e}^{\mathrm{i}\ell\phi} R(r), \tag{6}$$

and $R(r)$ is the radial eigenfunction written as

$$R_{n,\ell}(r) = \frac{1}{a^{|\ell|+1}}\left[\frac{(|\ell| + n)!}{2^{|\ell|}n!|\ell|!^2}\right] \times \mathrm{e}^{-\frac{r^2}{4a^2}} r^{|\ell|} F\left[-n, |\ell| + 1, \frac{r^2}{2a^2}\right], \tag{7}$$

where

$$a = \sqrt{\frac{1}{m\omega}} \tag{8}$$

is the magnetic length. Here, F is the degenerate hypergeometric function.

3 Landau levels analog for dipoles

Let us consider the non-relativistic limit of a single neutral spin-half particle with non-zero magnetic and electric dipole moments, moving in an external electromagnetic field [45]. In this limit, after neglecting terms of $O(\boldsymbol{E}^2)$ and $O(\boldsymbol{B}^2)$, the Anandan Hamiltonian looks like

$$H = -\frac{1}{2m}\big[\nabla - \mathrm{i}(\boldsymbol{\mu}\times\boldsymbol{E}) + \mathrm{i}(\boldsymbol{d}\times\boldsymbol{B})\big]^2 - \frac{\mu}{2m}\nabla\cdot\boldsymbol{E} + \frac{d}{2m}\nabla\cdot\boldsymbol{B}, \tag{9}$$

where $\boldsymbol{\mu}$ and $\boldsymbol{d}$ are the magnetic and electric dipole moments of the particle, and $\boldsymbol{B}$ and $\boldsymbol{E}$ are the magnetic and electric fields.

Under certain dipole–field configurations, effects analogous to standard Landau quantization occur. In this sense, we make use of the Aharonov–Casher and He–McKellar–Wilkens effects, in which neutral particles can interact with electric and magnetic fields through dipole moments. First, let us consider the case in which both d and $\boldsymbol{B}$ vanish in (9) and the Landau–Aharonov–Casher Hamiltonian takes the form

$$H = \frac{1}{2m}[\boldsymbol{p} - \mu\boldsymbol{A}_{\mathrm{AC}}]^2 - \frac{\mu}{2m}\nabla\cdot\boldsymbol{E}, \tag{10}$$

where the effective vector potential is given by

$$\boldsymbol{A}_{\mathrm{AC}} = \boldsymbol{n}\times\boldsymbol{E}, \quad \boldsymbol{n} = \frac{\boldsymbol{\mu}}{|\boldsymbol{\mu}|}, \tag{11}$$

and $\boldsymbol{n}$ is the unitary vector oriented in the dipole direction, so $\boldsymbol{\mu} = \mu\boldsymbol{n}$. We may define the associated field strength

$$\boldsymbol{B}_{\mathrm{AC}} = \nabla\times\boldsymbol{A}_{\mathrm{AC}}. \tag{12}$$

The exact field–dipole configuration for which the Landau–Aharonov–Casher effect takes place was demonstrated by Ericsson and Sjöqvist [19]. The conditions are a vanishing torque on the dipole, the electrostatical conditions $\partial_t\boldsymbol{E} = 0$, $\nabla\times\boldsymbol{E} = 0$ and B_{AC} should be uniform. Choosing $\boldsymbol{n}$ to be parallel to the z-axis, $\boldsymbol{n} = \hat{\mathrm{e}}_z$, one finds that the two first conditions are fulfilled if the electric field $\boldsymbol{E}$ is smooth and $E_z = 0$, and the particle moves in the x–y plane. So the condition $\nabla\cdot\boldsymbol{E} = 0$ reduces to the Gauss law and $\boldsymbol{B}_{\mathrm{AC}} = \rho\hat{\mathrm{e}}_z$, where ρ is a uniform volume charge density, fulfilling the third condition.

Now choosing the field configuration for the symmetric gauge as

$$\boldsymbol{E} = \frac{\rho}{2}r\hat{\mathrm{e}}_r, \tag{13}$$

we obtain the following effective potential:

$$\boldsymbol{A}_{\mathrm{AC}} = \frac{\rho}{2}r\hat{\mathrm{e}}_\phi. \tag{14}$$

Next we rewrite (10), making use of the cylindrical symmetry, in the form

$$H = \frac{1}{2m}\left[\boldsymbol{p} - \frac{m\omega}{2}r\hat{\mathrm{e}}_\phi\right]^2 - \frac{\omega}{2}, \tag{15}$$

where

$$\omega = \omega_{\mathrm{AC}} = \frac{\mu\rho}{m} \tag{16}$$

is the cyclotron frequency. Therefore, the Landau–Aharonov–Casher energy levels are given by [19]

$$\mathcal{E} = \left(n + \frac{|\ell| - \ell + 1}{2} - \frac{1}{2}\right)\omega_{\mathrm{AC}}, \tag{17}$$

where $n = 0, 1, 2, \ldots$. Thus, we have found energy levels for neutral magnetic polarized particles moving in an electric field in the same way as charged particles in a homogeneous magnetic field within the Landau quantization.

In the same way, in the case in which $\boldsymbol{\mu}$ and $\boldsymbol{E}$ vanishes, we may define another analogous effect. Writing the Landau–He–McKellar–Wilkens Hamiltonian as

$$H = \frac{1}{2m}[\boldsymbol{p} + d\boldsymbol{A}_{\mathrm{HMW}}]^2 + \frac{d}{2m}\nabla\cdot\boldsymbol{B}, \tag{18}$$

where

$$\boldsymbol{A}_{\mathrm{HMW}} = \boldsymbol{n}\times\boldsymbol{B}, \quad \boldsymbol{n} = \frac{\boldsymbol{d}}{|\boldsymbol{d}|}, \tag{19}$$

and $\boldsymbol{n}$ is the unitary vector oriented in the dipole direction, so $\boldsymbol{d} = d\boldsymbol{n}$, we may define the associated field strength as

$$\boldsymbol{B}_{\mathrm{HMW}} = \nabla\times\boldsymbol{A}_{\mathrm{HMW}}. \tag{20}$$

In this case we may also determine the field–dipole configuration for which the Landau–He–McKellar–Wilkens effect takes place. In the same way as within the Landau–Aharonov–Casher effect, in this case the torque on the dipole may vanish, $\partial_t\boldsymbol{B} = 0$, $\boldsymbol{B}$ must be smooth, and $\boldsymbol{B}_{\mathrm{HMW}}$ must be uniform. Thus, if $\boldsymbol{n} = \hat{\mathrm{e}}_z$, $B_z = 0$ and the particle moves in the x–y plane to fulfill the two first conditions. We may define $\nabla\cdot\boldsymbol{B} = \rho_{\mathrm{m}}$, where ρ_{m} is a uniform monopole magnetic volume density and $B_{\mathrm{HMW}} = \rho_{\mathrm{m}}\hat{\mathrm{e}}_z$.

Again choosing the symmetric gauge as

$$\boldsymbol{B} = \frac{\rho_{\mathrm{m}}}{2}r\hat{\mathrm{e}}_r, \tag{21}$$

we obtain the following effective potential:

$$\boldsymbol{A}_{\mathrm{AC}} = \frac{\rho_{\mathrm{m}}}{2}r\hat{\mathrm{e}}_\phi, \tag{22}$$

and rewrite (18) in the form

$$H = \frac{1}{2m}\left[\boldsymbol{p} + \frac{m\omega}{2}r\hat{\mathrm{e}}_\phi\right]^2 + \frac{\omega}{2}, \tag{23}$$

where

$$\omega = \omega_{\mathrm{HMW}} = \frac{d\rho_{\mathrm{m}}}{m} \tag{24}$$

is the cyclotron frequency. Therefore, the Landau–He–McKellar–Wilkens energy levels are given by [20]

$$\mathcal{E} = \left(n + \frac{|\ell| + \ell - 1}{2} + \frac{1}{2}\right)\omega_{\mathrm{HMW}}, \tag{25}$$

where $n = 0, 1, 2, \ldots$. So, we have found energy levels for neutral electric polarized particles in the same way as within Landau quantization for charged particles is a homogeneous magnetic field.

4 Non-commutative quantum mechanics

In quantum mechanics, several problems have been investigated in non-commutative space. Some interesting results are related to geometric phases [31–37, 45], and other effects which involve the dynamics of dipoles [43]. The idea consists in mapping non-commutative space onto commutative space by replacing the coordinates x^i and the momenta p^i [1] by Hermitian operators $\hat{x}^i$ and $\hat{p}^i$ which obey the relations

$$\begin{aligned} &[\hat{x}^i, \hat{x}^j] = \mathrm{i}\theta^{ij}, \\ &[\hat{p}^i, \hat{p}^j] = 0, \\ &[\hat{x}^i, \hat{p}^j] = \mathrm{i}\delta^{ij}, \end{aligned} \tag{26}$$

where $\theta^{ij} = \theta\epsilon^{ij}$ and ϵ^{ij} is the antisymmetric tensor. The time-independent Schrödinger equation in non-commutative space can be written in the form

$$H(x, p) \star \psi = \mathcal{E}\psi, \tag{27}$$

where $H(x, p)$ is the usual Hamiltonian, and the Moyal product is defined by

$$(f \star g)(x) = \exp\left(\frac{\mathrm{i}}{2}\theta^{ij}\partial_{x^i}\partial_{x^j}\right) f(x^i) g(x^j). \tag{28}$$

Here f and g are arbitrary functions. In non-commutative quantum mechanics, the Moyal product may be replaced by a Bopp shift [48], i.e., the $\star$-product may be changed into an ordinary product by replacing $H(x, p)$ by $H(\hat{x}, \hat{p})$ as follows:

$$H(\hat{x}^i, \hat{p}^i) = H\left(x^i - \frac{1}{2}\theta\epsilon^{ij} p^j, p^i\right), \tag{29}$$

where x^i and p^i are the generalized position and momentum coordinates in the usual quantum mechanics. Therefore, (29) is defined on commutative space and the effects due to non-commutativity may be calculated from the terms that contain the parameter θ. Thus, we must change the x^i in the Schrödinger equation by their Bopp shifts:

$$\hat{x}^i \to x^i - \frac{1}{2}\theta\epsilon^{ij} p^j. \tag{30}$$

Now we consider the case in which both space–space and momentum–momentum coordinates do not commute. The Bose–Einstein statistics in non-commutative quantum mechanics requires this kind of formulation [49, 50]. This is called phase space non-commutativity. Recently a series of physical systems was analyzed in non-commutative phase space [46, 47, 50]. In this case the operators x^i and p^i obey the commutation relations

$$\begin{aligned} &[\hat{x}^i, \hat{x}^j] = \mathrm{i}\theta^{ij}, \\ &[\hat{p}^i, \hat{p}^j] = \mathrm{i}\bar{\theta}^{ij}, \\ &[\hat{x}^i, \hat{p}^j] = \mathrm{i}\delta^{ij}, \end{aligned} \tag{31}$$

where $\bar{\theta}^{ij}$ is also an antisymmetric constant tensor, $\bar{\theta}^{ij} = \bar{\theta}\epsilon^{ij}$. Here, we have the Bopp shifts

$$H(\hat{x}^i, \hat{p}^i) = H\left(\lambda x^i - \frac{1}{2\lambda}\theta\epsilon^{ij} p^j, \lambda p^i + \frac{1}{2\lambda}\bar{\theta}\epsilon^{ij} x^j\right), \tag{32}$$

where the constant λ is a scaling factor. Thus, to map non-commutative phase space on commutative space, we must change $\hat{x}$ and $\hat{p}$ by their Bopp shifts:

$$\begin{aligned} &\hat{x}^i \to \lambda x^i - \frac{1}{2\lambda}\theta\epsilon^{ij} p^j, \\ &\hat{p}^i \to \lambda p^i + \frac{1}{2\lambda}\bar{\theta}\epsilon^{ij} x^j. \end{aligned} \tag{33}$$

The parameter λ, $\bar{\theta}$ and θ represent the non-commutative of the phase space, being related by the constraint [46, 47, 51]

$$\bar{\theta}^{ij}\theta^{ij} = \theta^{ij}\bar{\theta}^{ij} = \theta\bar{\theta}\mathbf{1} = 4\lambda^2(\lambda^2 - 1)\mathbf{1}, \tag{34}$$

where $\mathbf{1}$ is the identity matrix. In the following sections, we investigate the Landau-like quantization in non-commutative space and phase space. The standard Landau levels in non-commutative space are studied by Horvathy [40] and Gamboa et al. [42].

5 Landau–Aharonov–Casher levels in non-commutative space

In this section we analyze the Landau–Aharonov–Casher from the point of view of non-commutative quantum mechanics. In the case where d and $\boldsymbol{B}$ in (9) are vanishing, the

[1] Notice that x^i and p^i in the usual commutative space obey the following commutation relations $[x^i, x^j] = 0$; $[p^i, p^j] = 0$, $[x^i, p^j] = \mathrm{i}\delta^{ij}$.

Hamiltonian, for a magnetic dipole moment in the presence of an external electric field, is written in the form

$$H = -\frac{1}{2m}[\nabla - i\mu \boldsymbol{A}_{\mathrm{AC}}]^2 - \frac{\mu}{2m}\boldsymbol{\nabla}\cdot\boldsymbol{E}, \tag{35}$$

where

$$\boldsymbol{A}_{\mathrm{AC}} = \boldsymbol{n}\times\boldsymbol{E}, \qquad \boldsymbol{n} = \frac{\boldsymbol{\mu}}{|\boldsymbol{\mu}|}, \tag{36}$$

and $\boldsymbol{n}$ is unitary and oriented in the dipole direction.

Now we choose the dipole orientation along the z-axis, $\boldsymbol{n} = (0,0,1)$, and let the electric field be

$$\boldsymbol{E} = \frac{\rho}{2}(x, y, 0). \tag{37}$$

In this way, we obtain the following effective potential:

$$\boldsymbol{A}_{\mathrm{AC}} = \frac{\rho}{2}(-y, x, 0). \tag{38}$$

Thus, we rewrite (35) in the form

$$H = \frac{1}{2m}\left[\left(p_x + \frac{\mu\rho}{2}y\right)^2 + \left(p_y - \frac{\mu\rho}{2}x\right)^2\right] - \frac{\mu\rho}{2m}. \tag{39}$$

We can map the non-commutative space onto the commutative space via Bopp shifts, so the coordinates change as follows:

$$\begin{aligned} x &\to x - \frac{\theta}{2}p_y, \\ y &\to y + \frac{\theta}{2}p_x, \end{aligned} \tag{40}$$

and (39) takes the form

$$\begin{aligned} H = \frac{1}{2m}\Bigg[&\left(\left(1+\frac{\mu\rho\theta}{4}\right)p_x + \frac{\mu\rho}{2}y\right)^2 \\ &+ \left(\left(1+\frac{\mu\rho\theta}{4}\right)p_y - \frac{\mu\rho}{2}x\right)^2\Bigg] - \frac{\mu\rho}{2m}. \end{aligned} \tag{41}$$

We redefine the mass and frequency and obtain the following Hamiltonian:

$$\begin{aligned} H = \frac{1}{2\tilde{m}}&\left[\left(p_x + \frac{\tilde{m}\tilde{\omega}}{2}y\right)^2 + \left(p_y - \frac{\tilde{m}\tilde{\omega}}{2}x\right)^2\right] \\ &- \frac{\tilde{\omega}}{2}\left(1+\frac{\mu\rho\theta}{4}\right)^{-1}, \end{aligned} \tag{42}$$

where

$$\tilde{m} = \frac{m}{(1+\frac{\mu\rho\theta}{4})^2}, \qquad \tilde{\omega} = \frac{\mu\rho}{\tilde{m}(1+\frac{\mu\rho\theta}{4})}. \tag{43}$$

Thus, the non-commutative contributions modify the mass and the cyclotron frequency of the dipole.

To solve the Schrödinger equation, we may make use of the cylindrical symmetry of the system and rewrite the Hamiltonian (42) as

$$H = \frac{1}{2\tilde{m}}\left[\boldsymbol{p} - \frac{\tilde{m}\tilde{\omega}}{2}r\hat{\mathrm{e}}_\phi\right]^2 - \frac{\tilde{\omega}}{2}\left(1+\frac{\mu\rho\theta}{4}\right)^{-1}. \tag{44}$$

The Schrödinger equation in cylindrical coordinates is written in the form

$$\begin{aligned} &-\frac{1}{2\tilde{m}}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\phi^2}\right] - \frac{i\tilde{\omega}}{2}\frac{\partial\psi}{\partial\phi} \\ &\quad + \frac{\tilde{m}\tilde{\omega}^2}{8}r^2\psi - \frac{\tilde{\omega}}{2}\left(1+\frac{\mu\rho\theta}{4}\right)^{-1}\psi = \mathcal{E}\psi. \end{aligned} \tag{45}$$

We use the following ansatz to the solution of (45):

$$\psi = \mathrm{e}^{i\ell\phi}R(r), \tag{46}$$

where ℓ is an integer number. Thus, we can rewrite (45) as

$$\begin{aligned} &\frac{1}{2\tilde{m}}\left(R'' + \frac{1}{r}R' - \frac{\tilde{m}^2}{r^2}R\right) \\ &\quad + \left(\mathcal{E} - \frac{\tilde{m}\tilde{\omega}^2}{8}r^2 + \frac{\ell\tilde{\omega}}{2} + \frac{\tilde{\omega}}{2}\left(1+\frac{\mu\rho\theta}{4}\right)^{-1}\right)R = 0, \end{aligned} \tag{47}$$

and using the following change of variables:

$$\xi = \frac{\tilde{m}\tilde{\omega}}{2}r^2, \tag{48}$$

one can rewrite the radial Schrödinger equation in the form

$$\xi R'' + R' + \left(-\frac{\xi}{4} + \beta - \frac{\ell^2}{4\xi}\right)R = 0, \tag{49}$$

where

$$\beta = \frac{\mathcal{E}}{\tilde{\omega}} + \frac{\ell}{2} + \frac{1}{2}\left(1+\frac{\mu\rho\theta}{4}\right)^{-1}. \tag{50}$$

In the asymptotic limit of the solutions in (49), we may write the solution in the form

$$R(\xi) = \mathrm{e}^{-\xi/2}\xi^{|\ell|/2}\zeta(\xi). \tag{51}$$

So the hypergeometric equation is satisfied by the function $\zeta(\xi)$ given by

$$\zeta = F\left[-\left(\beta - \frac{|\ell|+1}{2}\right), |\ell|+1, \xi\right]. \tag{52}$$

The condition of finiteness of the function defined by (52) is that its first argument be a non-positive integer. Then the non-commutative Landau–Aharonov–Casher energy levels look like

$$\mathcal{E} = \left(n + \frac{|\ell| - \ell + 1}{2}\right)\tilde{\omega} - \frac{1}{2}\left(\frac{1+\mu\rho\theta}{4}\right)^{-1}\tilde{\omega}, \tag{53}$$

where $n = 0, 1, 2, \ldots$. The radial energy eigenfunctions are given by

$$R_{n,\ell}(r) = \frac{1}{\tilde{a}^{|\ell|+1}}\left[\frac{(|\ell|+n)!}{2^{|\ell|}n!|\ell|!^2}\right] \times \mathrm{e}^{-\frac{r^2}{4\tilde{a}^2}} r^{|\ell|} F\left[-n, |\ell|+1, \frac{r^2}{2\tilde{a}^2}\right], \quad (54)$$

where

$$\tilde{a} = \sqrt{\frac{1}{\tilde{m}\tilde{\omega}}} \quad (55)$$

is the new magnetic length redefined due the non-commutativity of the space coordinates.

If we enforce θ to vanish in (53), we recover the original result in (17). It is easy to see that the non-commutative corrections shift up the energy levels and reduce the magnetic length.

6 Landau–Aharonov–Casher in non-commutative phase space

In this section, we analyze the Landau–Aharonov–Casher problem in non-commutative phase space, using the description adopted in Sect. 4. To map non-commutative phase space into commutative space, we use the following Bopp shifts:

$$\begin{aligned} x &\to \lambda x - \frac{\theta}{2\lambda}p_y, \qquad p_x \to \lambda p_x + \frac{\bar{\theta}}{2\lambda}y, \\ y &\to \lambda y + \frac{\theta}{2\lambda}p_x, \qquad p_y \to \lambda p_y - \frac{\bar{\theta}}{2\lambda}x, \end{aligned} \quad (56)$$

where the scale factor λ is an arbitrary constant parameter. So (39) becomes

$$H = \frac{1}{2m}\left[\left(\left(\lambda + \frac{\mu\rho\theta}{4\lambda}\right)p_x + \frac{1}{2}\left(\mu\rho\lambda + \frac{\bar{\theta}}{\lambda}\right)y\right)^2 + \left(\left(\lambda + \frac{\mu\rho\theta}{4\lambda}\right)p_y - \frac{1}{2}\left(\mu\rho\lambda + \frac{\bar{\theta}}{\lambda}\right)x\right)^2\right] - \frac{\mu\rho}{2m}. \quad (57)$$

Performing a rescaling in the mass and frequency, we find

$$H = \frac{1}{2\tilde{m}}\left[\left(p_x + \frac{\tilde{m}\tilde{\omega}}{2}y\right)^2 + \left(p_y - \frac{\tilde{m}\tilde{\omega}}{2}x\right)^2\right] - \frac{\tilde{\omega}}{2\lambda}\left(\lambda + \frac{\mu\rho\theta}{4\lambda}\right)^{-1} + \frac{\bar{\theta}}{2\tilde{m}\lambda^2}\left(\lambda + \frac{\mu\rho\theta}{4\lambda}\right)^{-2}, \quad (58)$$

where

$$\tilde{m} = \frac{m}{(\lambda + \frac{\mu\rho\theta}{4\lambda})^2}, \qquad \tilde{\omega} = \frac{(\mu\rho\lambda + \frac{\bar{\theta}}{\lambda})}{\tilde{m}(\lambda + \frac{\mu\rho\theta}{4\lambda})}. \quad (59)$$

Here we have redefined the mass and cyclotron frequency in terms of the non-commutativity parameters, θ and $\bar{\theta}$. Making use of the cylindrical symmetry of the system, we can rewrite the Hamiltonian (58) in the form

$$H = \frac{1}{2\tilde{m}}\left[\boldsymbol{p} - \frac{\tilde{m}\tilde{\omega}}{2}r\hat{\mathrm{e}}_\phi\right]^2 - \frac{\tilde{\omega}}{2\lambda}\left(\lambda + \frac{\mu\rho\theta}{4\lambda}\right)^{-1} + \frac{\bar{\theta}}{2\tilde{m}\lambda^2}\left(\lambda + \frac{\mu\rho\theta}{4\lambda}\right)^{-2}. \quad (60)$$

The Schrödinger equation is therefore given by

$$-\frac{1}{2\tilde{m}}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\phi^2}\right] - \frac{\mathrm{i}\tilde{\omega}}{2}\frac{\partial\psi}{\partial\phi} + \frac{\tilde{m}\tilde{\omega}^2}{8}r^2\psi - \frac{\tilde{\omega}}{2\lambda}\left(\lambda + \frac{\mu\rho\theta}{4\lambda}\right)^{-1}\psi + \frac{\bar{\theta}}{2\tilde{m}\lambda^2}\left(\lambda + \frac{\mu\rho\theta}{4\lambda}\right)^{-2}\psi = \mathcal{E}\psi. \quad (61)$$

We use the following ansatz for the solution of (61)

$$\psi = \mathrm{e}^{\mathrm{i}\ell\phi}R(r), \quad (62)$$

and write the radial Schödinger equation as

$$\xi R'' + R' + \left(-\frac{\xi}{4} + \beta - \frac{\ell^2}{4\xi}\right)R = 0, \quad (63)$$

where we carried out change of variables

$$\xi = \frac{\tilde{m}\tilde{\omega}}{2}r^2, \quad (64)$$

and

$$\beta = \frac{\mathcal{E}}{\tilde{\omega}} + \frac{\ell}{2} + \frac{1}{2\lambda}\left(\lambda + \frac{\mu\rho\theta}{4\lambda}\right)^{-1} - \frac{\bar{\theta}}{2\tilde{m}\tilde{\omega}\lambda^2}\left(\lambda + \frac{\mu\rho\theta}{4\lambda}\right)^{-2}. \quad (65)$$

Studying the asymptotic limit of the solutions in (63), we can write the solution in the form

$$R(\xi) = \mathrm{e}^{-\xi/2}\xi^{|\ell|/2}\zeta(\xi). \quad (66)$$

So the hypergeometric equation is satisfied by the function $\zeta(\xi)$ which is given by

$$\zeta = F\left[-\left(\beta - \frac{|\ell|+1}{2}\right), |\ell+1|, \xi\right]. \quad (67)$$

The condition of finiteness of the function defined by Eq.(67) is that its first argument be a non-positive integer. Then the phase space non-commutative Landau–Aharonov–Casher energy levels are given by

$$\mathcal{E} = \left(n + \frac{|\ell| - \ell + 1}{2}\right)\tilde{\omega} - \frac{1}{2\lambda}\left(\lambda + \frac{\mu\rho\theta}{4\lambda}\right)^{-1}\tilde{\omega} + \frac{\bar{\theta}}{2\tilde{m}\lambda^2}\left(\lambda + \frac{\mu\rho\theta}{4\lambda}\right)^{-2}, \quad (68)$$

where $n = 0, 1, 2, \ldots$. Notice that, due to (34), we can express the energy levels (68), as well as $\tilde{m}$ and $\tilde{\omega}$, in terms of θ and $\bar{\theta}$:

$$\mathcal{E} = \left(n + \frac{|\ell| - \ell + 1}{2}\right)\tilde{\omega} - \frac{1}{2}\left(\frac{1}{2} \pm \frac{1}{2}\sqrt{1+\theta\bar{\theta}} + \frac{\mu\rho\theta}{4}\right)^{-1}\tilde{\omega} + \frac{\bar{\theta}}{2\tilde{m}}\left(\frac{1}{2} \pm \frac{1}{2}\sqrt{1+\theta\bar{\theta}} + \frac{\mu\rho\theta}{4}\right)^{-2}, \quad (69)$$

since $\lambda^2 = \frac{1}{2} \pm \frac{1}{2}\sqrt{1+\theta\bar{\theta}}$. The radial energy eigenfunctions are given by

$$R_{n,\ell}(r) = \frac{1}{\tilde{a}^{|\ell|+1}}\left[\frac{(|\ell|+n)!}{2^{|\ell|}n!|\ell|!^2}\right] \times \mathrm{e}^{-\frac{r^2}{4\tilde{a}^2}} r^{|\ell|} F\left[-n, |\ell|+1, \frac{r^2}{2\tilde{a}^2}\right], \quad (70)$$

where

$$\tilde{a} = \sqrt{\frac{1}{\tilde{m}\tilde{\omega}}} \quad (71)$$

is the redefined magnetic length. It is easy to show that we recover the case of non-commutative space as a special case of non-commutative phase space, taking $\lambda = 1$ and $\bar{\theta} = 0$. Also we recover the original effect if we make $\lambda = 1$, with the non-commutative parameters θ and $\bar{\theta}$ vanishing.

7 Landau–He–McKellar–Wilkens levels in non-commutative space

In the same way as for the study of the Landau–Aharonov–Casher levels, in the case that μ and $\boldsymbol{E}$ in (9) are equal to zero, we write the Hamiltonian for an electric dipole moment in the presence of a external magnetic field as

$$H = -\frac{1}{2m}[\nabla + \mathrm{i}d\boldsymbol{A}_{\mathrm{HMW}}]^2 + \frac{d}{2m}\nabla\cdot\boldsymbol{B}, \quad (72)$$

where

$$\boldsymbol{A}_{\mathrm{HMW}} = \boldsymbol{n}\times\boldsymbol{B}, \quad \boldsymbol{n} = \frac{\boldsymbol{d}}{|\boldsymbol{d}|}, \quad (73)$$

and $\boldsymbol{n}$ is unitary and oriented in the dipole direction.

Now we choose the dipole orientation in the direction of the z-axis, $\boldsymbol{n} = (0,0,1)$, and the magnetic field is chosen in the symmetric gauge

$$\boldsymbol{B} = \frac{\rho_{\mathrm{m}}}{2}(x, y, 0), \quad (74)$$

and we obtain the effective vector potential

$$\boldsymbol{A}_{\mathrm{HMW}} = \frac{\rho_{\mathrm{m}}}{2}(-y, x, 0), \quad (75)$$

where ρ_{m} is the magnetic monopole charge density. We rewrite (72) in the form

$$H = \frac{1}{2m}\left[\left(p_x - \frac{d\rho_{\mathrm{m}}}{2}y\right)^2 + \left(p_y + \frac{d\rho_{\mathrm{m}}}{2}x\right)^2\right] + \frac{d\rho_{\mathrm{m}}}{2m}. \quad (76)$$

We can map non-commutative space onto commutative space via Bopp shifts, so the coordinates change as follows:

$$x \to x - \frac{\theta}{2}p_y, \qquad y \to y + \frac{\theta}{2}p_x, \quad (77)$$

and (39) takes the form

$$H = \frac{1}{2m}\left[\left(\left(1 + \frac{d\rho_{\mathrm{m}}\theta}{4}\right)p_x - \frac{d\rho_{\mathrm{m}}}{2}y\right)^2 + \left(\left(1 + \frac{d\rho_{\mathrm{m}}\theta}{4}\right)p_y + \frac{d\rho_{\mathrm{m}}}{2}x\right)^2\right] + \frac{d\rho_{\mathrm{m}}}{2m}. \quad (78)$$

Performing a rescaling of the mass and the frequency, we obtain

$$H = \frac{1}{2\tilde{m}}\left[\left(p_x - \frac{\tilde{m}\tilde{\omega}}{2}y\right)^2 + \left(p_y + \frac{\tilde{m}\tilde{\omega}}{2}x\right)^2\right] + \frac{\tilde{\omega}}{2}\left(1 + \frac{d\rho_{\mathrm{m}}\theta}{4}\right)^{-1}, \quad (79)$$

where

$$\tilde{m} = \frac{m}{(1 + \frac{d\rho_{\mathrm{m}}\theta}{4})^2}, \qquad \tilde{\omega} = \frac{d\rho_{\mathrm{m}}}{\tilde{m}(1 + \frac{d\rho_{\mathrm{m}}\theta}{4})}. \quad (80)$$

So the non-commutative contributions modify the mass and cyclotron frequency of the dipole.

To solve the Schrödinger equation, we may make use of the cylindrical symmetry of the system and rewrite the Hamiltonian (79) thus:

$$H = \frac{1}{2\tilde{m}}\left[\boldsymbol{p} + \frac{\tilde{m}\tilde{\omega}}{2}r\hat{\mathrm{e}}_\phi\right]^2 + \frac{\tilde{\omega}}{2}\left(1 + \frac{d\rho_{\mathrm{m}}\theta}{4}\right)^{-1}. \quad (81)$$

The Schrödinger equation in cylindrical coordinates looks like

$$-\frac{1}{2\tilde{m}}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\phi^2}\right] - \frac{\mathrm{i}\tilde{\omega}}{2}\frac{\partial\psi}{\partial\phi} + \frac{\tilde{m}\tilde{\omega}^2}{8}r^2\psi + \frac{\tilde{\omega}}{2}\left(1 + \frac{d\rho_{\mathrm{m}}\theta}{4}\right)^{-1}\psi = \mathcal{E}\psi. \quad (82)$$

We use the following ansatz to the solution of (82):

$$\psi = e^{i\ell\phi} R(r), \tag{83}$$

where ℓ is an integer number. Hence, applying (82) presented as

$$\frac{1}{2\tilde{m}}\left(R'' + \frac{1}{r}R' - \frac{\tilde{m}^2}{r^2}R\right) + \left(\mathcal{E} - \frac{\tilde{m}\tilde{\omega}^2}{8}r^2 + \frac{\ell\tilde{\omega}}{2} - \frac{\tilde{\omega}}{2}\left(1 + \frac{d\rho_{\mathrm{m}}\theta}{4}\right)^{-1}\right)R = 0, \tag{84}$$

and using the following change of variables:

$$\xi = \frac{\tilde{m}\tilde{\omega}}{2}r^2, \tag{85}$$

one can rewrite the radial Schrödinger equation in the form

$$\xi R'' + R' + \left(-\frac{\xi}{4} + \beta - \frac{\ell^2}{4\xi}\right)R = 0, \tag{86}$$

where

$$\beta = \frac{\mathcal{E}}{\tilde{\omega}} + \frac{\ell}{2} - \frac{1}{2}\left(1 + \frac{d\rho_{\mathrm{m}}\theta}{4}\right)^{-1}. \tag{87}$$

Studying the asymptotic limit of the solutions in (86), we may write the solution as

$$R(\xi) = e^{-\xi/2}\xi^{|\ell|/2}\zeta(\xi). \tag{88}$$

So the hypergeometric equation is satisfied by the function $\zeta(\xi)$ of the form

$$\zeta = F\left[-\left(\beta - \frac{|\ell| + 1}{2}\right), |\ell| + 1, \xi\right]. \tag{89}$$

Again, the condition of finiteness of the function defined by (89) is that its first argument be a non-positive integer. Then the non-commutative Landau–He–McKellar–Wilkens energy levels are given by

$$\mathcal{E} = \left(n + \frac{|\ell| + \ell - 1}{2}\right)\tilde{\omega} + \frac{1}{2}\left(\frac{1 + d\rho_{\mathrm{m}}\theta}{4}\right)^{-1}\tilde{\omega}, \tag{90}$$

where $n = 0, 1, 2, \ldots$. The radial energy eigenfunctions are given by

$$R_{n,\ell}(r) = \frac{1}{\tilde{a}^{|\ell|+1}}\left[\frac{(|\ell| + n)!}{2^{|\ell|}n!|\ell|!^2}\right]e^{-\frac{r^2}{4\tilde{a}^2}}r^{|\ell|} \times F\left[-n, |\ell| + 1, \frac{r^2}{2\tilde{a}^2}\right], \tag{91}$$

where

$$\tilde{a} = \sqrt{\frac{1}{\tilde{m}\tilde{\omega}}} \tag{92}$$

is the magnetic length redefined by the non-commutative parameters.

If we make θ zero in (90), we recover the original effect given in (25). It is easy to see that the non-commutative corrections shift up the energy levels and reduce the magnetic length.

8 Landau–He–McKellar–Wilkens levels in non-commutative phase space

To map non-commutative phase space into commutative space, we use the following Bopp shifts:

$$x \to \lambda x - \frac{\theta}{2\lambda}p_y, \qquad p_x \to \lambda p_x + \frac{\bar{\theta}}{2\lambda}y,$$
$$y \to \lambda y + \frac{\theta}{2\lambda}p_x, \qquad p_y \to \lambda p_y - \frac{\bar{\theta}}{2\lambda}x, \tag{93}$$

where the scale factor λ is an arbitrary constant parameter. So (76) becomes

$$H = \frac{1}{2m}\left[\left(\left(\lambda + \frac{d\rho_{\mathrm{m}}\theta}{4\lambda}\right)p_x - \frac{1}{2}\left(d\rho_{\mathrm{m}}\lambda + \frac{\bar{\theta}}{\lambda}\right)y\right)^2 + \left(\left(\lambda + \frac{d\rho_{\mathrm{m}}\theta}{4\lambda}\right)p_y + \frac{1}{2}\left(d\rho_{\mathrm{m}}\lambda + \frac{\bar{\theta}}{\lambda}\right)x\right)^2\right] - \frac{d\rho_{\mathrm{m}}}{2m}. \tag{94}$$

Redefining the frequency and mass, we arrive at

$$H = \frac{1}{2\tilde{m}}\left[\left(p_x - \frac{\tilde{m}\tilde{\omega}}{2}y\right)^2 + \left(p_y + \frac{\tilde{m}\tilde{\omega}}{2}x\right)^2\right] + \frac{\tilde{\omega}}{2\lambda}\left(\lambda + \frac{d\rho_{\mathrm{m}}\theta}{4\lambda}\right)^{-1} - \frac{\bar{\theta}}{2\tilde{m}\lambda^2}\left(\lambda + \frac{d\rho_{\mathrm{m}}\theta}{4\lambda}\right)^{-2}, \tag{95}$$

where

$$\tilde{m} = \frac{m}{(\lambda + \frac{d\rho_{\mathrm{m}}\theta}{4\lambda})^2}, \qquad \tilde{\omega} = \frac{(d\rho_{\mathrm{m}}\lambda + \frac{\bar{\theta}}{\lambda})}{\tilde{m}(\lambda + \frac{d\rho_{\mathrm{m}}\theta}{4\lambda})}. \tag{96}$$

Making use of the cylindrical symmetry of the system, we can rewrite the Hamiltonian (95) in the form

$$H = \frac{1}{2\tilde{m}}\left[\boldsymbol{p} + \frac{\tilde{m}\tilde{\omega}}{2}r\hat{\mathrm{e}}_\phi\right]^2 + \frac{\tilde{\omega}}{2\lambda}\left(\lambda + \frac{d\rho_{\mathrm{m}}\theta}{4\lambda}\right)^{-1} - \frac{\bar{\theta}}{2\tilde{m}\lambda^2}\left(\lambda + \frac{d\rho_{\mathrm{m}}\theta}{4\lambda}\right)^{-2}. \tag{97}$$

The Schrödinger equation is given by

$$-\frac{1}{2\tilde{m}}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\phi^2}\right] - \frac{i\tilde{\omega}}{2}\frac{\partial\psi}{\partial\phi} + \frac{\tilde{m}\tilde{\omega}^2}{8}r^2\psi$$

$$+\frac{\tilde{\omega}}{2\lambda}\left(\lambda+\frac{d\rho_{\mathrm{m}}\theta}{4\lambda}\right)^{-1}\psi$$
$$-\frac{\bar{\theta}}{2\tilde{m}\lambda^2}\left(\lambda+\frac{d\rho_{\mathrm{m}}\theta}{4\lambda}\right)^{-2}\psi=\mathcal{E}\psi. \qquad (98)$$

We use the following ansatz to the solution of (98):

$$\psi=\mathrm{e}^{\mathrm{i}\ell\phi}R(r), \qquad (99)$$

and the radial Schödinger equation looks like

$$\xi R''+R'+\left(-\frac{\xi}{4}+\beta-\frac{\ell^2}{4\xi}\right)R=0, \qquad (100)$$

where we carried out the change of variables

$$\xi=\frac{\tilde{m}\tilde{\omega}}{2}r^2, \qquad (101)$$

and

$$\beta=\frac{\mathcal{E}}{\tilde{\omega}}+\frac{\ell}{2}-\frac{1}{2\lambda}\left(\lambda+\frac{d\rho_{\mathrm{m}}\theta}{4\lambda}\right)^{-1}$$
$$+\frac{\bar{\theta}}{2\tilde{m}\tilde{\omega}\lambda^2}\left(\lambda+\frac{d\rho_{\mathrm{m}}\theta}{4\lambda}\right)^{-2}. \qquad (102)$$

Studying the asymptotic limit of the solutions in (100), we can write the solution in the form

$$R(\xi)=\mathrm{e}^{-\xi/2}\xi^{|\ell|/2}\zeta(\xi). \qquad (103)$$

So the hypergeometric equation that is satisfied by the function $\zeta(\xi)$ is given by

$$\zeta=F\left[-\left(\beta-\frac{|\ell|+1}{2}\right),|\ell|+1,\xi\right]. \qquad (104)$$

The condition of finiteness of the function defined by (104) is that its first argument be a non-positive integer. Then the phase space non-commutative Landau–He–McKellar–Wilkens energy levels are given by

$$\mathcal{E}=\left(n+\frac{|\ell|+\ell-1}{2}\right)\tilde{\omega}+\frac{1}{2\lambda}\left(\lambda+\frac{d\rho\theta}{4\lambda}\right)^{-1}\tilde{\omega}$$
$$-\frac{\bar{\theta}}{2\tilde{m}\lambda^2}\left(\lambda+\frac{d\rho\theta}{4\lambda}\right)^{-2}, \qquad (105)$$

where $n=0,1,2,\ldots$. Note that based on (34), we can express the energy levels (105), as well as $\tilde{m}$ and $\tilde{\omega}$, in terms of θ and $\bar{\theta}$:

$$\mathcal{E}=\left(n+\frac{|\ell|+\ell-1}{2}\right)\tilde{\omega}$$
$$+\frac{1}{2}\left(\frac{1}{2}\pm\frac{1}{2}\sqrt{1+\theta\bar{\theta}}+\frac{d\rho\theta}{4}\right)^{-1}\tilde{\omega}$$
$$-\frac{\bar{\theta}}{2\tilde{m}}\left(\frac{1}{2}\pm\frac{1}{2}\sqrt{1+\theta\bar{\theta}}+\frac{d\rho\theta}{4}\right)^{-2}, \qquad (106)$$

since $\lambda^2=\frac{1}{2}\pm\frac{1}{2}\sqrt{1+\theta\bar{\theta}}$. The radial energy eigenfunctions are given by

$$R_{n,\ell}(r)=\frac{1}{\tilde{a}^{|\ell|+1}}\left[\frac{(|\ell|+n)!}{2^{|\ell|}n!|\ell|!^2}\right]\mathrm{e}^{-\frac{r^2}{4\tilde{a}^2}}r^{|\ell|}$$
$$\times F\left[-n,|\ell|+1,\frac{r^2}{2\tilde{a}^2}\right], \qquad (107)$$

where

$$\tilde{a}=\sqrt{\frac{1}{\tilde{m}\tilde{\omega}}} \qquad (108)$$

is the new magnetic length redefined by the non-commutative parameters. It is easy to show that we recover the case of non-commutative space as a special case of non-commutative phase space taking $\lambda=1$ and $\bar{\theta}=0$. Also we recover the original effect if we make $\lambda=1$ and the non-commutative parameters θ and $\bar{\theta}$ vanish.

9 Summary

We studied the Landau analog energy levels for neutral particles possessing non-zero magnetic and electric dipole moments in the presence of homogeneous electric and magnetic external fields in the context of non-commutative quantum mechanics. We analyzed Landau–Aharonov–Casher and Landau–He–McKellar–Wilkens quantization for magnetic and electric dipole, respectively. In both cases, we calculated the corrections in the Landau-like energy levels arising due to the non-commutativity in the space and phase space coordinates. We also found the corrections to the mass and cyclotron frequency in the non-commutative space and phase space, as well as the influence of non-commutativity in the energy levels, the radial wave functions and the magnetic length. Also, we verified that in the limit $\theta\to 0$ we recover the commutative result, and the case of NC space is a special case of NC phase space that we recover if we make $\lambda=1$ and $\bar{\theta}\to 0$.

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